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قائمة [5]

Discrete Fourier Transform (D.F.T)

If we want to analyze any signal, we must evaluate some parameters as:-

- 1- Freq. Content.
- 2- Power and Energy Density.
- 3- Amplitude.
- 4- Periodicity.

Both are analyzed in freq. domain

we want to convert signals from time domain to freq. domain, to study some parameters of the signals, as (freq. content and Power and Energy density), which can not be studied in time domain.

* Some applications depend on the analysis of the signals on freq. domain as filter design.

From Laplace Transform

$$x(t) \xrightarrow{\text{L.T.}} x(s)$$

time domain s-domain

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

- cont. fourier transform (FT)

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

↓ discretization

$X(K)$ Discrete Fourier transform
(DFT)

This doesn't work for digital systems

we need

Discretization

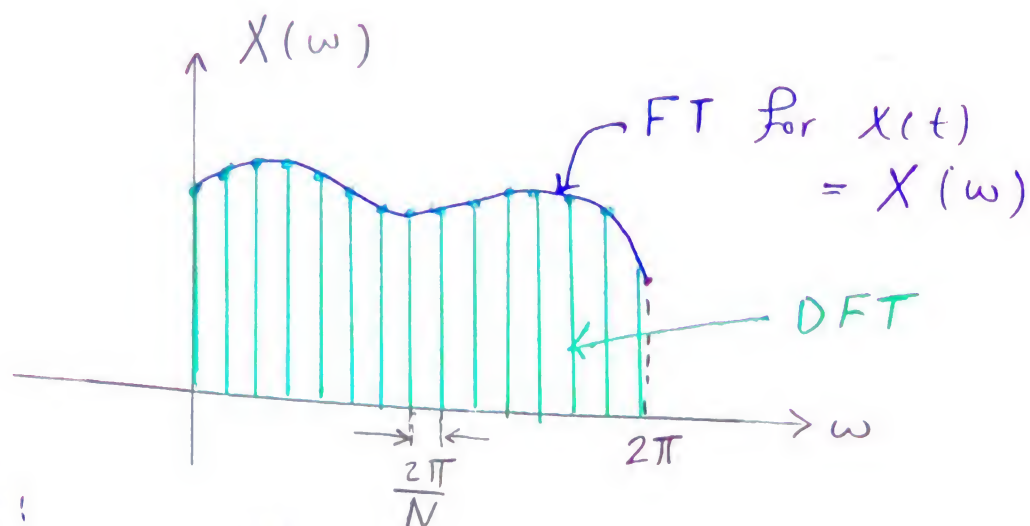
$$X(t) \xrightarrow{FT} X(\omega)$$

$$x(n) \xrightarrow{DFT} X(K)$$

* if $x(t)$ is periodic signal, \Leftarrow F.T we have
then the FT of $x(t)$ ($X(\omega)$) is also
Periodic. And it's periodic every
 2π interval of freq.

* if we assume the signals we deal with are
periodic.

⇒ Turn over



FT:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

assume $x(\omega)$ is discretized to N samples

$$\omega \longrightarrow \frac{2\pi}{N} K$$

K : the sample number for DFT.

$$K \Rightarrow [0 \rightarrow N-1]$$

N : No. of samples for DFT.

DFT:-

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} K n}$$

n : The sample no. for discrete time sequence $x(n)$, $n=0, 1, 2, \dots, N-1$

\Rightarrow Turn Over

Ex: $x(n) = \{0, 1, 2, 3\}$, find the 4-point DFT of $x(n)$, given $N=4$ $n=0, 1, 2, \dots, N-1$
 $k=0, 1, 2, \dots, N-1$

$$K=0 \Rightarrow X(0) = \sum_{n=0}^3 x(n)$$

$$= x(0) + x(1) + x(2) + x(3) = 6$$

$$K=1 \Rightarrow X(1) = \sum_{n=0}^3 x(n) e^{-j \frac{2\pi}{4} (1) \cdot n}$$

$$= x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 0 + e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j3\pi/2}$$

$$= \cancel{\cos(0)} - j \cancel{\sin(0)} + 2(\cancel{\cos(2\pi)} - j \cancel{\sin(2\pi)}) + 3(\cancel{\cos(3\pi/2)} - j \cancel{\sin(3\pi/2)})$$

$$= -2 + j2$$

$$K=2 \Rightarrow X(2) = \sum_{n=0}^3 x(n) e^{-j\pi n}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi}$$

$$= \cancel{\cos(\pi)} - j \cancel{\sin(\pi)} + 2(\cancel{\cos(2\pi)} - j \cancel{\sin(2\pi)}) + 3(\cancel{\cos(3\pi)} - j \cancel{\sin(3\pi)})$$

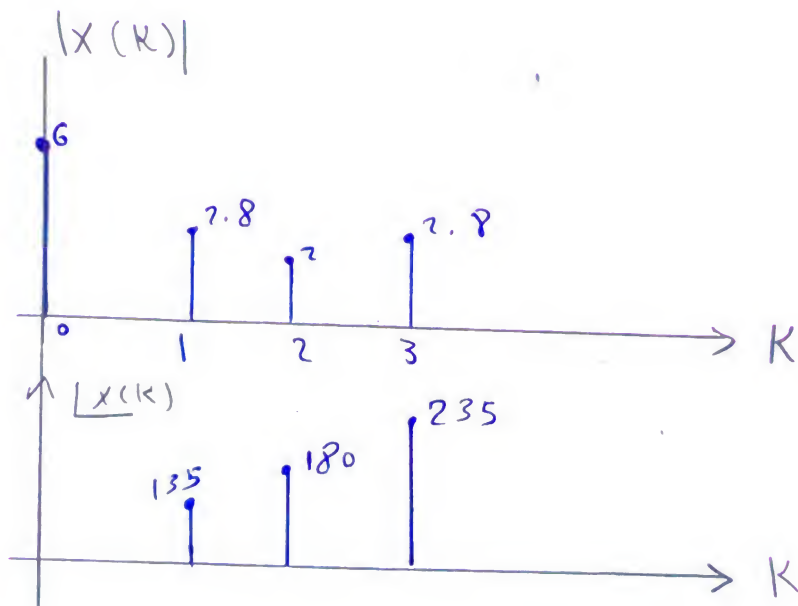
$$= -1 + 2 - 3 = -2$$

$$K=3 \Rightarrow X(3) = -2 - j2$$

$$X(K) = \{ 6, -2+j2, -2, -2-j2 \}$$

$$|X(K)| = \{ 6, 2\sqrt{2}, 2, 2\sqrt{2} \}$$

$$\angle X(K) = \{ 0, 135^\circ, 180^\circ, 225^\circ \}$$



Max K for exam 3 or 4

هناك طريقة أسرع من هذا Matrix

$$X(n) \xrightarrow{\text{DFT}} X(K) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi}{N} kn}$$

$$W_N = e^{-j \frac{2\pi}{N}}$$

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$K = 0, 1, 2, \dots, N-1$$

$$n = 0, 1, 2, \dots, N-1$$

$$X_N = \begin{pmatrix} x(k=0) \\ x(k=1) \\ \vdots \\ x(k=N-1) \end{pmatrix}$$

$$x_N = \begin{pmatrix} x(n=0) \\ x(n=1) \\ \vdots \\ x(n=N-1) \end{pmatrix}$$

$$X_N = \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} \left[w_N^{kn} \right] x_N$$

$N \times N$

Ex: $N=4$

$$\left[w_4^{kn} \right] = \begin{matrix} & \begin{matrix} n=0 & n=1 & n=2 & n=3 \end{matrix} \\ \begin{matrix} k=0 \\ k=1 \\ k=2 \\ k=3 \end{matrix} & \begin{pmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{pmatrix} \end{matrix}$$

$$w_4 = e^{-j \frac{2\pi}{4}} = \cos(90^\circ) - j \sin(90^\circ) = -j$$

$$w_4^0 = 1$$

$$w_4^2 = e^{-j\pi} = \cos(180^\circ) - j \sin(180^\circ) = -1$$

$$w_4^3 = e^{-j\frac{3\pi}{4}} = \cos(270^\circ) - j \sin(270^\circ) = j$$

$$w_4^4 = w_4^0 = 1$$

$$w_4^6 = w_4^2 = -1$$

$$w_4^9 = w_4^1 = -j$$

$$w_N^N = e^{-j \frac{2\pi N}{N}} = e^{-j2\pi} = 1$$

$$w_4^6 = w_4^2 \cdot w_4^4$$

$$w_4^9 = w_4^4 \cdot w_4^4 \cdot w_4^1$$

\Rightarrow Turn over

$$[w_4] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix}$$

DFT for $x(n)$: ($N=4$)

$$X_4 = [w_4] x_4$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{pmatrix}$$

$\uparrow x(K=0)$
 $\downarrow x(K=4)$

For $N=3$

$$[w_3] = \begin{matrix} & \begin{matrix} n=0 & n=1 & n=2 \end{matrix} \\ \begin{matrix} K=0 \\ K=1 \\ K=2 \end{matrix} & \begin{pmatrix} w_3^0 & w_3^0 & w_3^0 \\ w_3^0 & w_3^1 & w_3^2 \\ w_3^0 & w_3^2 & w_3^4 \end{pmatrix} \end{matrix}$$

3x3

$$w_3^0 = 1$$

$$w_3^1 = e^{-j\frac{2\pi}{3}} = j$$

$$w_3^2 = e^{-j\frac{4\pi}{3}} = -1$$

$$w_3^4 = w_3^1 = j$$

$$[w_3] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & -1 \\ 1 & -1 & j \end{pmatrix}$$

Twiddle

Factor

Ex: $x(n) = \{0, 5, 1\}$, find the 3-point DFT

$$N=3$$

$$\text{DFT } X(K) \Rightarrow X_3 = [W_3] X_3$$

$$X_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & -1 \\ 1 & -1 & j \end{pmatrix} \begin{pmatrix} 0.5 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.5 \\ 0.5+j \\ 0.5 \end{pmatrix}$$

$$X(K) = \{1.5, 0.5+j, -0.5\}$$

$$|X(K)| = \{1.5, 1.11, 0.5\}$$

$$\angle X(K) = \{0, 63.4, 180^\circ\}$$

